On integrable contact systems and bi-Hamiltonian structures

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Symplectic geometry

- Symplectic manifolds are the natural geometric frameworks for Hamiltonian mechanics.
- Let me recall that a symplectic manifold (M, ω) is a $2n$-dimensional manifold endowed with a 2-form ω such that $\mathrm{d}\omega = 0$ and $\omega^n \neq 0$.
- The Hamiltonian vector field X_h of a function $h \in \mathscr{C}^\infty(M)$ is given by $\omega(X_h, \cdot) = 0$.
- In a neighborhood of each point in M there are canonical (or Darboux) coordinates (q^i, p_i) in which

$$\omega = \mathrm{d}q^i \wedge \mathrm{d}p_i \,, \quad X_h = \frac{\partial h}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial h}{\partial q^i}\frac{\partial}{\partial p_i} \,.$$

Liouville–Arnol'd theorem

Theorem (Liouville–Arnol'd)

Let $f_1, \dots, f_n$ be independent functions in involution (i.e., $\{f_i, f_j\} = 0\ \forall i, j$) on a symplectic manifold (M^{2n}, ω). Let $M_\Lambda = \{x \in M \mid f_i = \Lambda_i\}$ be a regular level set.

1. *Any compact connected component of M_Λ is diffeomorphic to $\mathbb{T}^n$.*
2. *On a neighborhood of M_Λ there are coordinates (φ^i, J_i) such that*

$$\omega = \mathrm{d}\varphi^i \wedge \mathrm{d}J_i\,,$$

and $f_i = f_i(J_1, \dots, J_n)$, so the Hamiltonian vector fields read

$$X_{f_i} = \frac{\partial f_i}{\partial J_j} \frac{\partial}{\partial \varphi^j}\,.$$

Liouville–Arnol'd theorem

Corollary

Let (M^{2n}, ω, h) be a Hamiltonian system. Suppose that $f_1, \ldots, f_n$ are independent conserved quantities (i.e. $X_h(f_i) = 0 \; \forall \, i$) in involution. Then, on a neighborhood of M_Λ there are Darboux coordinates (φ^i, J_i) such that $h = h(J_1, \ldots, J_n)$, so the Hamiltonian dynamics are given by

$$\frac{\mathrm{d}\varphi^i}{\mathrm{d}t} = \frac{\partial h}{\partial J_i} \frac{\partial}{\partial \varphi^i} \, ,$$

$$\frac{\mathrm{d}J_i}{\mathrm{d}t} = 0 \, .$$

Problem

Given a Hamiltonian system (M^{2n}, ω, h), we would like to find n independent conserved quantities in involution $f_1, \dots, f_n$, in order to construct action-angle coordinates (φ^i, J_i).

Magri *et al.* developed a method for constructing such conserved quantities by computing the eigenvalues of a $(1, 1)$-tensor field N verifying certain compatibility conditions.

Compatible Poisson structures

Definition

Let M be a manifold. Two Poisson tensors are Λ and Λ_1 on M are said to be **compatible** if $\Lambda + \Lambda_1$ is also a Poisson tensor on M.

Definition

A vector field $X \in \mathfrak{X}(M)$ is called **bi-Hamiltonian** if it is a Hamiltonian vector field w.r.t. two compatible Poisson structures, namely,

$$X = \Lambda(\mathrm{d}h, \cdot) = \Lambda_1(\mathrm{d}h_1, \cdot)\,,$$

for two functions $h, h_1 \in \mathscr{C}^\infty(M)$.

Poisson – Nijehuis structures

- The linear map $\sharp_\Lambda \colon \mathrm{T}^*_x M \ni \alpha \mapsto \Lambda(\alpha, \cdot) \in \mathrm{T}_x M$ is an isomorphism iff Λ comes from a symplectic structure ω. In that case, $\sharp_\omega := \sharp_\Lambda^{-1}(v) = \iota_v \omega$.
- In that situation, we can define the $(1,1)$-tensor field

$$N = \sharp_{\Lambda_1} \circ \sharp_\Lambda^{-1} .$$

Poisson – Nijehuis structures

Theorem (Magri and Morosi, 1984)

Let (M, ω) be a symplectic manifold and Λ_1 a bivector. Consider the $(1, 1)$-tensor field

$$N = \sharp_{\Lambda_1} \circ \sharp_{\omega}^{-1} .$$

If Λ_1 is a Poisson tensor compatible with Λ, then the Nijehuis torsion T_N of N vanishes. In that case, the eigenvalues of N are in involution w.r.t. both Poisson brackets.
The pair (Λ, N) is called a ***Poisson – Nijenhuis structure*** *on M.*

Poisson – Nijehuis structures

Corollary

If a vector field $X \in \mathfrak{X}(M)$ is bi-Hamiltonian w.r.t. to ω and Λ_1 (i.e., $X = \sharp_\omega \mathrm{d}h = \sharp_{\Lambda_1} \mathrm{d}h_1$), then the eigenvalues of N form a family of conserved quantities in involution w.r.t. both Poisson brackets.

Proposition (Magri *et al.*, 1997)

Let (Λ, N) *be a Poisson–Nijenhuis structure on* M. *Consider the functions*

$$I_k = \frac{1}{k} \operatorname{Tr} N^k \,, \quad k \in \{1, \ldots, n\} \,.$$

In a neighbourhood of a point $x \in M$ *such that* $\mathrm{d}I_1(x) \wedge \cdots \wedge \mathrm{d}I_n(x) \neq 0$ *there are coordinates* (λ^i, μ_i) *which are canonical both for* Λ *and* N, *namely,*

$$\Lambda = \frac{\partial}{\partial \lambda^i} \wedge \frac{\partial}{\partial \mu_i} \,,$$
$$N^* \mathrm{d}\lambda^i = \lambda^i \mathrm{d}\lambda^i \,,$$
$$N^* \mathrm{d}\mu_i = \lambda^i \mathrm{d}\mu_i \,.$$

Contact geometry

Definition

A (co-oriented) **contact manifold** is a pair (M, η), where M is an $(2n+1)$-dimensional manifold and η is a 1-form on M such that the map

$$\begin{aligned} \flat_\eta \colon \mathfrak{X}(M) &\to \Omega^1(M) \\ X &\mapsto \iota_X \mathrm{d}\eta + \eta(X)\eta, \end{aligned}$$

is an isomorphism of $\mathscr{C}^\infty(M)$-modules.

- There exists a unique vector field $\mathcal{R}$ on (M, η), called the **Reeb vector field**, given by $\mathcal{R} = \flat_\eta^{-1}(\eta)$, or, equivalently,

$$\iota_\mathcal{R} \mathrm{d}\eta = 0, \quad \iota_\mathcal{R}\eta = 1.$$

Contact geometry

- The **Hamiltonian vector field** of $f \in \mathscr{C}^\infty(M)$ is given by

$$X_f = \flat_\eta^{-1}(\mathrm{d}f) - (\mathcal{R}(f) + f)\,\mathcal{R}\,,$$

- Around each point on M there exist **Darboux coordinates** (q^i, p_i, z) such that

$$\eta = \mathrm{d}z - p_i \mathrm{d}q^i,$$
$$\mathcal{R} = \frac{\partial}{\partial z},$$
$$X_f = \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} - \left(\frac{\partial f}{\partial q^i} + p_i \frac{\partial f}{\partial z}\right)\frac{\partial}{\partial p_i} + \left(p_i \frac{\partial f}{\partial p_i} - f\right)\frac{\partial}{\partial z}.$$

Contact Hamiltonian systems

Definition

A **contact Hamiltonian system** (M, η, h) is a co-oriented contact manifold (M, η) with a fixed **Hamiltonian function** $h \in \mathscr{C}^\infty(M)$.

- The dynamics of (M, η, h) is determined by the integral curves of the Hamiltonian vector field X_h of h w.r.t. η.

Contact Hamiltonian systems

- In Darboux coordinates, these curves $c(t) = (q^i(t), p_i(t), z(t))$ are determined by the **contact Hamilton equations**:

$$
\begin{aligned}
\frac{\mathrm{d}q^i(t)}{\mathrm{d}t} &= \frac{\partial h}{\partial p_i} \circ c(t)\,,\\
\frac{\mathrm{d}p_i(t)}{\mathrm{d}t} &= -\frac{\partial h}{\partial q^i} \circ c(t) - p_i(t)\frac{\partial h}{\partial z} \circ c(t)\,,\\
\frac{\mathrm{d}z(t)}{\mathrm{d}t} &= p_i(t)\frac{\partial h}{\partial p_i} \circ c(t) - h \circ c(t)\,.
\end{aligned}
$$

Example (The harmonic oscillator with linear damping)

Consider the solution $x\colon \mathbb{R} \to \mathbb{R}$ of the second-order ordinary differential equation

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2}(t) = -x(t) - \kappa \frac{\mathrm{d}x}{\mathrm{d}t}(t)\,,$$

where $\kappa \in \mathbb{R}$. Defining $p = \mathrm{d}x/\mathrm{d}t$, we can reduce it to the system of first-order ordinary differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) = p(t)\,, \quad \frac{\mathrm{d}p}{\mathrm{d}t}(t) = -x(t) - \kappa p(t)\,.$$

We can obtain this system as the two first contact Hamilton equations from the contact Hamilton system $(\mathbb{R}^3, \eta, h)$, where $\eta = \mathrm{d}z - p\mathrm{d}x$ and

$$h = \frac{p^2}{2} + \frac{x^2}{2} + \kappa z\,.$$

Jacobi manifolds

Definition

A **Jacobi structure** on a manifold M is a pair (Λ, E) where Λ is a bivector and E a vector field such that the composition rule $\{\cdot,\cdot\}$ on $\mathscr{C}^\infty(M)$ given by

$$\{f, g\} = \Lambda(\mathrm{d}f, \mathrm{d}g) + fE(g) - gE(f)\,,$$

is a Lie bracket, called the **Jacobi bracket**. The triple (M, Λ, E) is called a **Jacobi manifold**.

In particular, $\{\cdot,\cdot\}$ is a Poisson bracket iff $E \equiv 0$.

Jacobi structure of a contact manifold

A contact manifold (M, η) is endowed with a Jacobi bracket determined by

$$\{f, g\} = X_f(g) + g\mathcal{R}(f)\,.$$

Contact ≅ homogeneous symplectic

Remark

The most natural and efficient way to extend the theory of integrable systems (such as the Liouville–Arnold theorem, Magri's results, and others) to the realm of contact geometry is to utilise the equivalence between the categories of contact manifolds and homogeneous symplectic manifolds.

Contact ≅ homogeneous symplectic

Remark

The notions and assumptions will be the same as for integrable systems in the usual sense, with the additional requirement of homogeneity in the corresponding structures. By projecting, we will derive the equivalent results in the contact category.

Exact symplectic manifolds

Definition

An **exact symplectic manifold** is a pair (M, θ), where θ is a **symplectic potential** on M, i.e., $\omega = -\mathrm{d}\theta$ is a symplectic form on M. The **Liouville vector field** $\nabla \in \mathfrak{X}(M)$ is given by

$$\iota_{\nabla}\omega = -\theta\,.$$

A tensor field A on P is called k-homogeneous (for $k \in \mathbb{Z}$) if

$$\mathcal{L}_{\nabla} A = kA\,.$$

Trivial symplectization of a co-oriented contact manifold

Definition

Let (M, η) be a co-oriented contact manifold. Then, the trivial bundle $\pi_1 \colon M^{\mathrm{symp}} = M \times \mathbb{R}_+ \to M$, $\pi_1(x, r) = x$ can be endowed with the symplectic potential $\theta(x, r) = r\eta(x)$. The Liouville vector field reads $\nabla = r\partial_r$.
We will refer to $(M^{\mathrm{symp}}, \theta)$ as the **trivial symplectization** of (M, η).

Trivial symplectization of a co-oriented contact manifold

Proposition

There is a one-to-one correspondence between functions $f(x)$ *on* M *and* 1*-homogeneous functions* $f^{\mathrm{symp}}(x, r) = -rf(x)$ *on* M^{symp} *such that the symplectic* $X_{f^{\mathrm{symp}}}$ *and contact* X_f *Hamiltonian vector fields are related as follows:*

$$\mathrm{T}\pi_1\left(X_{f^{\mathrm{symp}}}\right) = X_f \,.$$

Moreover, the Poisson $\{\cdot,\cdot\}_\theta$ *and Jacobi* $\{\cdot,\cdot\}_\eta$ *brackets have the correspondence*

$$\{f^{\mathrm{symp}}, g^{\mathrm{symp}}\}_\omega = \left(\{f, g\}_\eta\right)^{\mathrm{symp}} .$$

The non co-orientable case

Theorem (Grabowski, 2013)

There is a canonical one-to-one correspondence between contact distributions $C \subset \mathsf{T}M$ on M and symplectic $\mathbb{R}^\times$-principal bundles $\pi \colon P \to M$ over M.
More precisely, the symplectic $\mathbb{R}^\times$-principal bundle associated with C is $(C^\circ)^\times = C^\circ \setminus 0_{\mathsf{T}^*M} \subset \mathsf{T}^*M$ (i.e., the annihilator of C with the zero section removed), whose symplectic form is the restriction to $(C^\circ)^\times$ of the canonical symplectic form ω_M on T^*Q. It is called the ***symplectic cover*** *of (M, C).*

Homogeneous integrable system

Definition

A **homogeneous integrable system** consists of an exact symplectic manifold (M^{2n}, θ) and a map $F = (f_1, \ldots, f_n) \colon M \to \mathbb{R}^n$ such that the functions $f_1, \ldots, f_n$ are independent, in involution and homogeneous of degree 1 (w.r.t. the Liouville vector field ∇ of θ) on a dense open subset $M_0 \subseteq M$. We will denote it by (M, θ, F).

For simplicity's sake, in this talk I will assume that $M_0 = M$.

- Consider the exact symplectic manifold (M, θ), with Liouville vector field ∇.
- Around each point in M, there are canonical coordinates (q^i, p_i) where $\theta = p_i \mathrm{d}q^i$.
- Then, a straightforward computation shows that $\nabla = p_i \frac{\partial}{\partial p_i}$.
- Note that coordinates may be canonical for $\omega = -\mathrm{d}\theta$ but not for θ. For instance, in the coordinates $\tilde{q}^i = q^i$, $\tilde{p}_i = p_i + e^{q_i}$ we have

$$\theta = \sum_i (\tilde{p}_i - e^{\tilde{q}^i}) \mathrm{d}\tilde{q}^i \,, \quad \omega = \mathrm{d}\tilde{q}^i \wedge \mathrm{d}\tilde{p}_i \,, \quad \nabla = \left(\tilde{p}_i - e^{\tilde{q}^i}\right) \frac{\partial}{\partial \tilde{p}_i} \,.$$

- In particular, the Liouville–Arnol'd theorem provides coordinates which are canonical for ω, but not necessarily for θ or ∇.

Homogeneous Liouville–Arnol'd theorem

Theorem (Colombo, de León, Lainz, L. G., 2023)

Let (M, θ, F) be a homogeneous integrable system with $F = (f_1, \dots, f_n)$. Given $\Lambda \in \mathbb{R}^n$, suppose that $M_\Lambda = F^{-1}(\Lambda)$ is connected, and assume that, in an open neighbourhood U of M_Λ, the Hamiltonian vector fields X_{f_i} are complete, $\operatorname{rank} \mathrm{T}F = n$ *and $F \colon U \to V = F(U)$ is a trivial bundle. Then, $U \cong \mathbb{T}^k \times \mathbb{R}^{n-k} \times V$ and there is a chart $(\hat{U} \subseteq U; y^i, A_i)$ of M s.t.*

1. *$A_i = M_i^j f_j$, where M_i^j are homogeneous functions of degree* 0 *depending only on $f_1, \dots, f_n$,*
2. $\theta = A_i \mathrm{d} y^i$,
3. *$X_{f_i} = N_i^j \frac{\partial}{\partial y^j}$, with (N_i^j) the inverse matrix of (M_i^j).*

Theorem (Magri's theorem for exact symplectic manifolds)

Let (Λ, N) be a Poisson–Nijenhuis structure on M such that $\Lambda = \omega^{-1}$ for an exact symplectic structure $\omega = -\mathrm{d}\theta$. Consider the functions

$$I_k = \frac{1}{k} \operatorname{Tr} N^k\,, \quad k \in \{1, \dots, n\}\,.$$

In a neighbourhood of a point $x \in M$ such that $\mathrm{d}I_1(x) \wedge \cdots \wedge \mathrm{d}I_n(x) \neq 0$ there are coordinates (λ^i, μ_i) which are canonical both for θ and N, namely,

$$\theta = \mu_i \mathrm{d}\lambda^i\,,$$
$$N^* \mathrm{d}\lambda^i = \lambda^i \mathrm{d}\lambda^i\,, \quad N^* \mathrm{d}\mu_i = \lambda^i \mathrm{d}\mu_i\,.$$

Moreover, $\mathcal{L}_\nabla \lambda^i = 0$ and $\mathcal{L}_\nabla \mu_i = \mu_i$, where ∇ is the Liouville vector field w.r.t. θ.

Definition

A **completely integrable contact system** is a triple (M, η, F), where (M^{2n+1}, η) is a co-oriented contact manifold and $F = (f_0, \ldots, f_n) \colon M \to \mathbb{R}^{n+1}$ is a map such that

1. $f_0, \ldots, f_n$ are in involution, i.e., $\{f_\alpha, f_\beta\} = 0 \; \forall \, \alpha, \beta \in \{0, \ldots, n\}$,
2. rank $\mathrm{T}F \geq n$ on a dense open subset $M_0 \subseteq M$.

Proposition

Let (M, η) be a co-oriented contact manifold and $F \colon M \to \mathbb{R}^{n+1}$ a smooth map. Consider the trivial symplectization, i.e., $M^{\mathrm{symp}} = M \times \mathbb{R}_+$ endowed with the symplectic potential $\theta(x, r) = r\eta(x)$, and the map $F^{\mathrm{symp}}(x, r) = -rF(x)$. Then, $(M^{\mathrm{symp}}, \theta, F^{\mathrm{symp}})$ is a homogeneous integrable system iff (M, η, F) is a completely integrable contact system.

Jacobi–Nijehuis structures $\rightsquigarrow$ action-angle coordinates

- Let (M, η) be a contact manifold.
- Consider its trivial symplectization $(M \times \mathbb{R}_+, \theta = r\eta)$, and let Λ denote the Poisson tensor defined by $\omega = -\mathrm{d}\theta$.
- By construction, θ (resp. Λ) is homogeneous of degree 1 (resp. -1).
- Suppose that Λ_1 is a second Poisson tensor compatible with Λ and homogeneous of degree -1.
- Then, $N = \sharp_{\Lambda_1} \circ \sharp_{\Lambda}^{-1}$ is Nijehuis and homogeneous of degree 0.

Jacobi–Nijehuis structures $\rightsquigarrow$ action-angle coordinates

- Utilizing Magri's theorem, we can find canonical coordinates (λ_i, μ_i) such that

$$\theta = \mu_i \mathrm{d}\lambda^i\,, \quad N^* \mathrm{d}\lambda^i = \lambda^i \mathrm{d}\lambda^i\,, \quad N^* \mathrm{d}\mu_i = \lambda^i \mathrm{d}\mu_i\,.$$

- The coordinates λ^i are eigenvalues of N^*, and hence homogeneous of degree 0.
- The coordinates μ_i are homogeneous of degree 1, by the homogeneity of θ.

Jacobi–Nijehuis structures $\rightsquigarrow$ action-angle coordinates

- Unhomogeneizing, we have $2n+2$ functions in M:

$$\overline{\lambda}^i = \lambda^i \circ \pi_M\,, \quad \overline{\mu}_i = \frac{\mu_i}{r} \circ \pi_M\,,$$

where $\pi_M \colon M \times \mathbb{R}_+ \to M$ is the canonical projection and r the global coordinate of $\mathbb{R}_+$.

- We have $(n+1)$ functions in involution w.r.t. the Jacobi bracket:

$$\{\overline{\mu}_i, \overline{\mu}_j\}_\eta = 0\,.$$

- Moreover, they lead to coordinates $(\overline{\lambda}^i, \tilde{\mu}_i)$ on M, where $\tilde{\mu}_i = -\frac{\overline{\mu}_i}{\overline{\mu}_j}$ for $i \in \{0, \ldots, n\} \setminus \{j\}$.

Jacobi–Nijehuis structures $\rightsquigarrow$ action-angle coordinates

- In these coordinates,

$$\eta = \mathrm{d}\overline{\lambda}^{j} - \sum_{i \neq j} \tilde{\mu}_i \mathrm{d}\overline{\lambda}^{i}\,,$$

$$X_{\overline{\mu}_i} = \frac{\partial}{\partial \overline{\lambda}^{i}}\,.$$

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Eskerrik asko!

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