Hybrid dynamical systems for the modelling of rigid bodies with impacts

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Hybrid systems

- A hybrid dynamical system is one which combines continuous and discrete transitions.
- The dynamics of such systems are continuous "most of the time", except at some instants at which abrupt changes occur.
- This framework may be used to model mechanical systems with impacts.

Hybrid systems

Definition

A **hybrid system** is a 4-tuple $\mathscr{H} = (M, X, S, \Delta)$, formed by

1. a manifold M,
2. a vector field $X \in \mathfrak{X}(M)$,
3. a submanifold $S \subset M$ of codimension 1 or greater,
4. an embedding $\Delta \colon S \to M$.

The dynamics generated by $\mathscr{H}$ are the curves $c \colon I \subseteq \mathbb{R} \to M$ such that

$$\begin{aligned} \dot{c}(t) &= X(c(t))\,, && \text{if } c(t) \notin S\,, \\ c^+(t) &= \Delta(c^-(t))\,, && \text{if } c(t) \in S\,, \end{aligned}$$

where

$$c^{\pm}(t) = \lim_{\tau \to t^{\pm}} c(\tau)\,.$$

Hybrid Hamiltonian systems

Definition

A hybrid dynamical system (M, X, S, Δ) is said to be a **hybrid Hamiltonian system** and denoted by $\mathscr{H}_H$ if

1. $M \subseteq \mathrm{T}^*Q$ is a zero-codimensional submanifold of the cotangent bundle T^*Q of a manifold Q,
2. S projects onto a codimension-one submanifold $\pi_Q(S)$ of Q,
3. $\pi_Q \circ \Delta = \pi_Q$,
4. $X = X_H$ is the Hamiltonian vector field of $H \in \mathscr{C}^\infty(\mathrm{T}^*Q)$ w.r.t. ω_Q.

Hybrid Hamiltonian systems

Physically,

- Q represents the space of positions,
- T^*Q the phase space,
- X_H the dynamics between the impacts,
- $\pi_Q(S)$ the hypersurface where impacts occur, and
- Δ the change of momenta on the impacts.

Hybrid Lie group action

Definition

A Lie group action $\Phi\colon G \times Q \to Q$ is called a **hybrid action for $\mathscr{H}_H$** if its cotangent lift $\Phi^{\mathrm{T}^*}\colon G \times \mathrm{T}^*Q \to \mathrm{T}^*Q$ satisfies the following conditions:

1. H is Φ^{T^*}-invariant, namely, $H \circ \Phi_g^{\mathrm{T}^*} = H$ for all $g \in G$,
2. the restriction $\Phi^{\mathrm{T}^*}\big|_{G\times S}$ is a Lie group action of G on S,
3. the impact map is equivariant w.r.t. this action, i.e.,

$$\Delta \circ \Phi_g^{\mathrm{T}^*}\Big|_S = \Phi_g^{\mathrm{T}^*} \circ \Delta\,, \quad \forall\, g \in G\,.$$

Hybrid momentum map

Definition

Let $\Phi\colon G \times Q \to Q$ be a hybrid action for $\mathscr{H}_H$. A momentum map $\mathbf{J}\colon \mathrm{T}^*Q \to \mathfrak{g}^*$ for the cotangent lift action Φ^{T^*} is called a **generalized hybrid momentum map** if, for each connected component $C \subseteq S$ and for each regular value μ^- of $\mathbf{J}$, there is another regular value μ^+ such that

$$\Delta(\mathbf{J}|_C{}^{-1}(\mu^-)) \subset \mathbf{J}^{-1}(\mu^+)\,.$$

In particular, if $\mu^- = \mu^+$ it is called a **hybrid momentum map**.
A **hybrid regular value** of $\mathbf{J}$ is a regular value of both $\mathbf{J}$ and $\mathbf{J}|_S$.

Hybrid momentum map

In other words, $\mathbf{J}$ is a generalized hybrid momentum map if, for every point in the connected component C of the switching surface S such that the momentum before the impact takes a value of μ^-, the momentum will take a value μ^+ after the impact.

It is a hybrid momentum map if its value does not change with the impacts.

Hybrid reduction

- There is a natural action of a Lie group G on the dual $\mathfrak{g}^*$ of its Lie algebra, called the coadjoint action.
- The **isotropy subgroup** G_μ at $\mu \in \mathfrak{g}^*$ is the Lie subgroup given by those elements of G whose coadjoint action leaves μ invariant, namely,

$$G_\mu = \{g \in G \mid g \cdot \mu = \mu\} \ .$$

- In the case of an Abelian Lie group, $G_\mu = G$.

Hybrid reduction

Proposition

If μ^- and μ^+ are regular values of $\mathbf{J}$ such that $\Delta\left(\mathbf{J}|_S{}^{-1}(\mu^-)\right) \subset \mathbf{J}^{-1}(\mu^+)$, then the isotropy subgroups in μ^- and μ^+ coincide, that is, $G_{\mu^-} = G_{\mu^+}$.

Hybrid reduction

Theorem (Colombo, de León, Eyrea Irazú, and L. G., 2022)

Let $\Phi\colon G \times Q \to Q$ be a hybrid action on $\mathscr{H}_H$. Assume that G is connected and that $\Phi^{\mathsf{T}^*}\colon G \times \mathsf{T}^*Q \to \mathsf{T}^*Q$ is free and proper. Consider a sequence $\{\mu_i\}_{i\in I\subseteq \mathbb{N}}$ of hybrid regular values of $\mathbf{J}$, such that $\Delta\left(\mathbf{J}|_S{}^{-1}(\mu_i)\right) \subset \mathbf{J}^{-1}(\mu_{i+1})$. Let $G_{\mu_i} = G_{\mu_0}$ be the isotropy subgroup in μ_i under the co-adjoint action. Then, the reduction leads to a sequence of reduced hybrid forced Hamiltonian systems

$$\mathscr{H}_H^{\mu_i} = \left(\mathbf{J}^{-1}(\mu_i)/G_{\mu_0},\, X_{H_{\mu_i}}, \mathbf{J}|_S{}^{-1}(\mu_i)/G_{\mu_0}, (\Delta)_{\mu_i}\right).$$

Hybrid reduction

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & \mathbf{J}^{-1}(\mu_i) & \hookleftarrow & \mathbf{J}|_S{}^{-1}(\mu_i) & \xrightarrow{\Delta|_{\mathbf{J}^{-1}(\mu_i)}} & \mathbf{J}^{-1}(\mu_{i+1}) & \hookleftarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \dfrac{\mathbf{J}^{-1}(\mu_i)}{G_{\mu_0}} & \hookleftarrow & \mathbf{J}|_S{}^{-1}(\mu_i)/G_{\mu_0} & \xrightarrow{(\Delta)_{\mu_i}} & \dfrac{\mathbf{J}^{-1}(\mu_{i+1})}{G_{\mu_0}} & \hookleftarrow & \cdots
\end{array}
$$

Nonholomic systems

- Roughly speaking, a nonholonomic constraint is a constraint in the velocities which cannot be reduced to a constraint in the positions.
- Geometrically, this is expressed by the fact that the phase space is a (co)distribution of the (co)tangent bundle.
- Let $L\colon \mathrm{T}Q \to \mathbb{R}$ be a mechanical Lagrangian function, namely,

$$L(q, v) = \frac{1}{2} g_q(v, v) - V(q)\,,$$

where g is a Riemannian metric on Q.
- The Hamiltonian counterpart of L is $H\colon \mathrm{T}^*Q \to \mathbb{R}$, where

$$H(q, p) = \frac{1}{2} g_q^{-1}(p, p) + V(q)\,.$$

Nonholomic systems

- Suppose that the system is subject to the (linear) nonholonomic constraints given by the distribution

$$D = \{ v \in \mathsf{T}Q \mid \mu^a(v) = 0,\ a = 1, \ldots, k \} ,$$

where $\mu^a = \mu^a_i(q)\mathrm{d}q^i$ are constraint one-forms.

- Denote by $C = \flat_g(D)$ the associated codistribution.
- The **nonholonomic vector field** X_H^{nh} of H is given by

$$\iota_{X_H^{\mathrm{nh}}}\omega_Q = \mathrm{d}H - \lambda_a \, \mu^a \, ,$$

with the constraint

$$\mathsf{T}\pi_Q \left(X_H^{\mathrm{nh}} \right) \in \Gamma(D) \, .$$

- Here, $\omega_Q = \mathrm{d}q^i \wedge \mathrm{d}p_i$ denotes the canonical symplectic form, and λ_a are Lagrange multipliers.

Example: Rolling disk hitting walls

- Consider a homogeneous circular disk of radius R and mass m moving freely in the plane.
- The configuration space is $Q = \mathbb{R}^2 \times \mathbb{S}^1$.
- The Hamiltonian function $H \colon \mathrm{T}^*Q \to \mathbb{R}$ of the system is

$$H = \frac{1}{2m}(p_x^2 + p_y^2) + \frac{1}{2mk^2} p_\theta^2 \,,$$

where $(x, y, \theta, p_x, p_y, p_\theta)$ are the bundle coordinates in $\mathrm{T}^*(\mathbb{R}^2 \times \mathbb{S}^1)$.
- The coords. (x, y) represent then position of the center of the disk, and θ represents the angle between a fixed reference point of the disk and the y-axis.
- Here m is the mass of the disk and k is a constant such that mk^2 is the moment of inertia of the disk.

Example: Rolling disk hitting walls

- There are two rough walls situated at $y = 0$ and at $y = h > R$.
- Assume that the impact with a wall is such that the disk rolls without sliding and that the change of the velocity along the y-direction is characterized by an elastic constant e.
- The switching surface is $S = C_1 \cup C_2$, where

$$C_1 = \{(x, y, \vartheta, p_x, p_y, p_\vartheta) \mid y = R,\ p_x = Rp_\vartheta/k^2 \text{ and } p_y < 0\}\,,$$
$$C_2 = \{(x, y, \vartheta, p_x, p_y, p_\vartheta) \mid y = h - R,\ p_x = Rp_\vartheta/k^2 \text{ and } p_y > 0\}\,,$$

and the impact map $\Delta\colon S \to \mathrm{T}^*Q$ is given by

$$\Delta\colon \left(p_x^-, p_y^-, p_\theta^-\right) \mapsto \left(\frac{R^2 p_x^- + Rp_\theta^-}{k^2 + R^2}, -ep_y^-, k^2 \frac{Rp_x^- + p_\theta^-}{k^2 + R^2}\right)$$

Example: Rolling disk hitting walls

- The condition $p_x = Rp_\vartheta / k^2$ comes from the nonholonomic constraint of the walls.
- The conditions on the sign of p_y ensure that the y-component of the momenta points towards corresponding the wall.

Example: Rolling disk hitting walls

- Consider polar coordinates (r, φ) in the plane, namely,

$$x = r\cos\varphi\,, \quad y = r\sin\varphi\,.$$

- Let $(r, \varphi, \theta, p_r, p_\varphi, p_\theta)$ be the induced bundle coordinates in T^*Q.

Example: Rolling disk hitting walls

- In these coordinates,

$$
\begin{aligned}
H &= \frac{1}{2m} p_r^2 + \frac{1}{2mr^2} p_\varphi^2 + \frac{1}{2mk^2} p_\theta^2 \,, \\
C_1 &= \left\{ r \sin\varphi = R \,,\; p_r \cos\varphi - \frac{p_\varphi \sin\varphi}{r} = \frac{R p_\theta}{k^2} \right. \\
&\qquad \left. \text{and } \; p_r \sin\varphi + \frac{p_\varphi \cos\varphi}{r} < 0 \right\} , \\
C_2 &= \left\{ r \sin\varphi = h - R \,,\; p_r \cos\varphi - \frac{p_\varphi \sin\varphi}{r} = \frac{R p_\theta}{k^2} \right. \\
&\qquad \left. \text{and } \; p_r \sin\varphi + \frac{p_\varphi \cos\varphi}{r} > 0 \right\} .
\end{aligned}
$$

Example: Rolling disk hitting walls

- The impact map $\Delta\colon (p_r^-, p_\varphi^-, p_\theta^-) \mapsto (p_r^+, p_\varphi^+, p_\theta^+)$ is given by

$$
\begin{aligned}
p_r^+ &= (2\cos^2\varphi - 1)p_r^- - 2\sin\varphi\cos\varphi\frac{p_\varphi^-}{r}\,,\\
p_\varphi^+ &= -p_\varphi^-\,,\\
p_\theta^+ &= p_\theta^-\,.
\end{aligned}
$$

- Consider the Lie group action

$$
\begin{aligned}
\Phi\colon \mathbb{T}^2 \times Q &\to Q\\
(\alpha, \beta; r, \varphi, \theta) &\mapsto (r, \varphi + \alpha, \theta + \beta)\,.
\end{aligned}
$$

- It is clear that H is invariant under the cotangent lift action $\Phi^{\mathrm{T}^*}\colon \mathbb{T}^2 \times \mathrm{T}^*Q \to \mathrm{T}^*Q$.

Example: Rolling disk hitting walls

- The associated momentum map is $\mathbf{J} = (p_\varphi, p_\theta)$.
- Notice that it is a generalized hybrid momentum map but not a hybrid momentum map, namely, $\Delta\left(\mathbf{J}|_C{}^{-1}(\mu^-)\right) \subset \mathbf{J}^{-1}(\mu^+)$ but $\mathbf{J}^{-1}(\mu^+) \neq \mathbf{J}^{-1}(\mu^-)$.
- Let $\mu = (\mu_\varphi, \mu_\theta)$ be a hybrid regular value of $\mathbf{J}$.

Example: Rolling disk hitting walls

- The reduced connected components of the switching surface can be written as

$$C_{1,\mu^-} = \left\{ r\sin\gamma = R\,,\ \ p_r\cos\gamma - \frac{\mu_\varphi \sin\varphi}{r} = \frac{R\mu_\theta}{k^2} \right.$$
$$\left. \text{and } p_r\sin\varphi + \frac{\mu_\varphi\cos\gamma}{r} < 0 \text{ for some } \gamma \in [0, 2\pi) \right\},$$
$$C_{2,\mu^-} = \left\{ r\sin\gamma = h - R\,,\ \ p_r\cos\gamma - \frac{\mu_\varphi \sin\gamma}{r} = \frac{R\mu_\theta}{k^2} \right.$$
$$\left. \text{and } p_r\sin\gamma + \frac{\mu_\varphi\cos\gamma}{r} > 0 \text{ for some } \gamma \in [0, 2\pi) \right\}.$$

Example: Rolling disk hitting walls

- The reduced impact map reads

$$\Delta_{\mu^-} : p_r^- \mapsto (2\cos^2\gamma - 1)p_r^- - 2\sin\gamma\cos\gamma\frac{\mu_\varphi^-}{r},$$

where γ is determined by the relation between v_r^-, μ_φ^- and μ_θ^+.

Integrable hybrid Hamiltonian systems

- A particular case of hybrid reduction is when we have the Abelian Lie group action $\Phi\colon \mathbb{R}^n \times \mathrm{T}^*Q \to \mathrm{T}^*Q$ generated by the Hamiltonian flows of n functions $f_1, \ldots, f_n$ in involution.
- In that case, we can identify the momentum map with $F = (f_1, \ldots, f_n)\colon \mathrm{T}^*Q \to \mathbb{R}^n$.
- We may obtain action-angle coordinates for each time interval between impacts. The action-angle coordinates before and after the impact will be related by Δ.

Liouville–Arnol'd theorem

Theorem (Liouville–Arnol'd)

Let $f_1, \ldots, f_n$ be independent functions in involution (i.e., $\{f_i, f_j\} = 0 \ \forall i, j$) on a symplectic manifold (M^{2n}, ω). Let $M_\Lambda = \{x \in M \mid f_i = \Lambda_i\}$ be a regular level set.

1. *Any compact connected component of M_Λ is diffeomorphic to $\mathbb{T}^n$.*
2. *On a neighborhood of M_Λ there are coordinates (φ^i, J_i) such that*

$$\omega = \mathrm{d}\varphi^i \wedge \mathrm{d}J_i \,,$$

and $f_i = f_i(J_1, \ldots, J_n)$, so the Hamiltonian vector fields read

$$X_{f_i} = \frac{\partial f_i}{\partial J_j} \frac{\partial}{\partial \varphi^j} \,.$$

Liouville–Arnol'd theorem

Corollary

Let (M^{2n}, ω, h) *be a Hamiltonian system. Suppose that* $f_1, \ldots, f_n$ *are independent conserved quantities (i.e.* $X_h(f_i) = 0 \ \forall\, i$*) in involution. Then, on a neighborhood of* M_Λ *there are Darboux coordinates* (φ^i, J_i) *such that* $H = H(J_1, \ldots, J_n)$*, so the Hamiltonian dynamics are given by*

$$\frac{\mathrm{d}\varphi^i}{\mathrm{d}t} = \frac{\partial H}{\partial J_i}\frac{\partial}{\partial \varphi^i}\,,$$

$$\frac{\mathrm{d}J_i}{\mathrm{d}t} = 0\,.$$

Definition

Let (M, S, X, Δ) be a hybrid dynamical system. A function $f : M \to \mathbb{R}$ is called a **generalized hybrid constant of the motion** if

1. $Xf = 0$,
2. For each connected component $C \subseteq S$ and each $a \in \operatorname{Im} f$, there exists a $b \in \operatorname{Im} f$ such that

$$\Delta\left(f|_C{}^{-1}(a)\right) \subseteq f^{-1}(b)\,.$$

In particular, f is called a **hybrid constant of the motion** if, in addition, $b = a$ for each $a \in \operatorname{Im} f$.

Definition

Let Q be an n-dimensional manifold. A **completely integrable hybrid Hamiltonian system** is a 5-tuple $(\mathrm{T}^*Q, S, X_H, \Delta, F)$, formed by a hybrid Hamiltonian system $(\mathrm{T}^*Q, S, X_H, \Delta)$, together with a function $F = (f_1, \ldots, f_n) \colon \mathrm{T}^*Q \to \mathbb{R}^n$ such that:

1. $\operatorname{rank} \mathrm{T}_x F = n \quad$ a.e.,
2. the functions $f_1, \ldots, f_n$ are generalized hybrid constant of the motion
3. $\{f_i, f_j\} = X_{f_j}(f_i) = 0 \quad \forall\, i, j \in \{1, \ldots, n\}$.

Theorem (L. G. and Colombo, 2024)

Consider a completely integrable hybrid Hamiltonian system (T^*Q, S, X_H, Δ), *with* $F = (f_1, \ldots, f_n)$, *where* $n = \dim Q$. *Let* M_Λ *be a regular level set of* F. *Then:*

1. *For each regular level set* M_Λ *and each connected component* $C \subseteq S$, *there exists a* $\Lambda' \in \mathbb{R}^n$ *such that* $\Delta(M_\Lambda \cap C) \subset M_{\Lambda'} = F^{-1}(\Lambda')$.
2. *On a neighbourhood* U_λ *of* M_Λ *there are coordinates* (φ^i, s_i) *s.t.*
   1. $\omega_Q = \mathrm{d}\varphi^i \wedge \mathrm{d}s_i$,
   2. *the action coordinates* s_i *are functions depending only on the integrals* $f_1, \ldots, f_n$,
   3. *the continuous part hybrid dynamics are given by*

   $$\dot{\varphi}^i = \Omega^i(s_1, \ldots, s_n), \qquad \dot{s}_i = 0\,.$$

   4. *In these coordinates, for each connected component* $C \subseteq S$, *the impact map reads* $\Delta \colon (\varphi^i_-, s_i^-) \in M_\Lambda \cap C \mapsto (\varphi^i_+, s_i^+) \in M_{\Lambda'}$, *where* $s_1^+, \ldots, s_n^+$ *are functions depending only on* $s_1^-, \ldots, s_n^-$.

Rolling disk with a harmonic potential hitting fixed walls

- Consider the example from before with the addition of an oscillatory potential to the Hamiltonian function:

$$H = \frac{1}{2m}(p_x^2 + p_y^2) + \frac{1}{2mk^2} p_\theta^2 + \frac{1}{2}\Omega^2(x^2 + y^2)\,.$$

- Recall that the switching surface is $S = C_1 \cup C_2$, where

$$C_1 = \left\{ \left(x, R, \theta, p_x, p_y, \frac{k^2}{R} p_x \right) \mid x, p_x, p_y \in \mathbb{R},\ \theta \in \mathbb{S}^1 \right\},$$

$$C_2 = \left\{ \left(x, h - R, \theta, p_x, p_y, \frac{k^2}{R} p_x \right) \mid x, p_x, p_y \in \mathbb{R},\ \theta \in \mathbb{S}^1 \right\},$$

Rolling disk with a harmonic potential hitting fixed walls

and the impact map $\Delta\colon S \to \mathrm{T}^*Q$ is given by

$$\left(p_x^-, p_y^-, p_\theta^-\right) \mapsto \left(\frac{R^2 p_x^- + k^2 R p_\theta^-}{k^2 + R^2}, -e p_y^-, \frac{R p_x^- + k^2 p_\theta^-}{k^2 + R^2}\right) .$$

Rolling disk with a harmonic potential hitting fixed walls

- For simplicity's sake, let us hereafter take $m = R = k = \Omega = 1$.
- The functions

$$f_1 = \frac{p_x^2 + x^2}{2}, \quad f_2 = \frac{p_y^2 + y^2}{2}, \quad f_3 = \frac{p_\theta^2}{2},$$

  are conserved quantities with respect to the Hamiltonian dynamics of H.
- Moreover, $\{f_i, f_j\} = 0$ and $\mathrm{d}f_1 \wedge \mathrm{d}f_2 \wedge \mathrm{d}f_3 \neq 0$ a.e.
- Let $F = (f_1, f_2, f_3)\colon \mathrm{T}^*(\mathbb{R}^2 \times \mathbb{S}) \to \mathbb{R}^3$.
- It is clear that, for $\Lambda \neq 0$, the level sets $F^{-1}(\Lambda)$ are diffeomorphic to $\mathbb{S} \times \mathbb{S} \times \mathbb{R}$.

Rolling disk with a harmonic potential hitting fixed walls

- In the intersection of their domains of definition, the functions

$$\phi^1 = \arctan\left(\frac{x}{p_x}\right), \quad \phi^2 = \arctan\left(\frac{y}{p_y}\right), \quad \phi^3 = \frac{\theta}{p_\theta}$$

  are coordinates on each level set $F^{-1}(\Lambda)$ for $\Lambda \neq 0$.
- Additionally, $\omega_Q = \mathrm{d}\phi^i \wedge \mathrm{d}f_i$.
- In these coordinates, the Hamiltonian function reads

$$H = f_1 + f_2 + f_3 \,.$$

- Hence, its Hamiltonian vector field is simply

$$X_H = \frac{\partial}{\partial \phi^1} + \frac{\partial}{\partial \phi^2} + \frac{\partial}{\partial \phi^3} \,.$$

Rolling disk with a harmonic potential hitting fixed walls

- In the action-angle coordinates (ϕ^i, f_i), the connected components of the impact surface read

$$C_1 = \left\{ \left(\phi^i, f_i\right) \mid 2 f_2 \sin^2 \phi^2 = R^2 \text{ and } f_3 = \frac{2k^4 f_1 \cos^2 \phi^1}{R^2} \right\},$$
$$C_2 = \left\{ \left(\phi^i, f_i\right) \mid 2 f_2 \sin^2 \phi^2 = (h - R)^2 \text{ and } f_3 = \frac{2k^4 f_1 \cos^2 \phi^1}{R^2} \right\}.$$

Rolling disk with a harmonic potential hitting fixed walls

- The relations between the coordinates before, (ϕ^i_-, f_i^-), and after, (ϕ^i_+, f_i^+), are

$$\phi^1_+ = \phi^1_- \,, \qquad \phi^2_+ = -\arctan\left(\frac{\tan\phi^2_-}{e}\right) \,, \qquad \phi^3_+ = \phi^3_- \,,$$

$$f_1^+ = f_1^- \,, \qquad f_2^+ = e^2 f_2 + \tfrac{1-e^2}{2} a^2 \,, \qquad f_3^+ = f_3^- \,,$$

where $a = R$ or $a = h - R$ depending on the wall where the impact takes place.

Hamilton – Jacobi equation

Consider a Hamiltonian function $h\colon \mathsf{T}^*Q \to \mathbb{R}$. Given a closed one-form $\gamma \in \Omega^1(Q)$, the following assertions are equivalent:

1. γ is a solution of the **Hamilton – Jacobi (HJ) equation**

$$\gamma^* \mathrm{d}h = 0\,,$$

2. the following diagram is commutative:

$$\begin{array}{ccc} \mathsf{T}^*Q & \xrightarrow{\;X_h\;} & \mathsf{T}\mathsf{T}^*Q \\ \gamma \uparrow \; \downarrow \pi_Q & & \mathsf{T}\pi_Q \downarrow \; \uparrow \mathsf{T}\gamma \\ Q & \xrightarrow[\;X_h^\gamma\;]{} & \mathsf{T}Q \end{array}\,,$$

3. $c\colon I \subseteq \mathbb{R} \to Q$ integral curve of $X_h^\gamma \Longrightarrow \gamma \circ c$ integral curve of X_h;
4. X_h is tangent to $\operatorname{Im} \gamma$.

Hybrid HJ equation

Definition

Let $\mathscr{H}_h = (\mathrm{T}^*Q, X_h, S, \Delta)$ be a hybrid Hamiltonian system. A **solution of the Hamilton – Jacobi (HJ) problem** for $\mathscr{H}_h$ is a sequence $\{\gamma_i\}_i$ of closed one-forms $\gamma_i \in \Omega^1(Q)$ such that:

1. each γ_i is a solution of the HJ equation for h, namely, $\gamma_i^* \mathrm{d}h = 0$;
2. they satisfy the compatibility condition

$$\operatorname{Im}(\Delta \circ \gamma_i) \subset \operatorname{Im} \gamma_{i+1} .$$

Hybrid HJ equation

Theorem (Clark, 2020)

Consider a hybrid Hamiltonian system $\mathscr{H}_H = (M, X_H, S, \Delta)$. Let $\{\gamma_i\}_i$ be a sequence of closed one-forms $\gamma_i \in \Omega^1(Q)$. Then, the following statements are equivalent:

1. *The sequence $\{\gamma_i\}_i$ is a solution of the hybrid HJ problem for $\mathscr{H}_h$.*
2. *For every continuous and piecewise smooth curve $c \colon \mathbb{R} \to Q$ s.t.*
    1. *c intersects $\pi_Q(S)$ at $\{t_i\}_i$,*
    2. *c satisfies the equations*

$$\dot{c}(t) = T\pi_Q \circ X_H \circ \gamma_i \circ c(t), \qquad t_i < t < t_{i+1},$$
$$\gamma_{i+1} \circ c(t_{i+1}) = \Delta \circ \gamma_i \circ c(t_{i+1}),$$

the curve $\tilde{c} \colon \mathbb{R} \to \mathrm{T}^*Q$ given by $\tilde{c}(t) = \gamma_i \circ c(t)$ for $t \in [t_i, t_{i+1})$ is an integral curve of the hybrid dynamics.

Example: Rolling disk hitting walls

- Consider the example from the reduction section:

$$H = \frac{1}{2m}(p_x^2 + p_y^2) + \frac{1}{2mk^2} p_\theta^2\,,$$
$$C_1 = \{(x, y, \vartheta, p_x, p_y, p_\vartheta) \mid y = R,\ p_x = R p_\vartheta / k^2 \text{ and } p_y < 0\}\,,$$
$$C_2 = \{(x, y, \vartheta, p_x, p_y, p_\vartheta) \mid y = h - R,\ p_x = R p_\vartheta / k^2 \text{ and } p_y > 0\}\,,$$
$$\Delta \colon \left(p_x^-, p_y^-, p_\theta^-\right) \mapsto \left(\frac{R^2 p_x^- + R p_\theta^-}{k^2 + R^2}, -e p_y^-, k^2 \frac{R p_x^- + p_\theta^-}{k^2 + R^2}\right).$$

Example: Rolling disk hitting walls

- A general solution of the HJ equation for H is

$$\gamma_i = a_i \mathrm{d}x + b_i \mathrm{d}y + c_i \mathrm{d}y \,,$$

where a_i, b_i, c_i are constants.

- The relation between these constants before and after an impact is determined by the compatibility condition:

$$a_{i+1} = \frac{R^2 a_i + R c_i}{k^2 + R^2}, \; b_{i+1} = -e b_i, \text{ and } c_{i+1} = k^2 \frac{R a_i + c_i}{k^2 + R^2}.$$

- The initial values (a_0, b_0, c_0) correspond with the initial values $(p_x(0), p_y(0), p_\vartheta(0))$ of the momenta at time zero.

Example: Rolling disk hitting walls

- Each one-form γ_i determines a Lagrangian submanifold of $\mathsf{T}^*(\mathbb{R}^2 \times \mathbb{S}^1)$, namely,

$$\operatorname{Im}\gamma_i = \left\{(x, y, \vartheta, p_x, p_y, p_\vartheta) \in \mathsf{T}^*(\mathbb{R}^2 \times \mathbb{S}^1) \mid p_x = a_i,\ p_y = b_i,\ p_\vartheta = c_i\right\}$$

Theorem (Ohsawa and Bloch, 2009)

Assume that D *is a completely nonholonomic distribution, that is,*

$$\mathsf{T}Q = \langle \{D, [D, D], [D, [D, D]], \ldots\} \rangle \ .$$

Let γ *be a one-form on* Q *such that* $\operatorname{Im}\gamma \subset C$ *and* $\mathrm{d}\gamma(v, w) = 0$ *for any* $v, w \in \Gamma(D)$*. Then, the following statements are equivalent:*

1. *For every integral curve* c *of* $T\pi_Q \circ X_H \circ \gamma$*, the curve* $\gamma \circ c$ *is an integral curve of* X_H^{nh}*.*
2. *The one-form* γ *satisfies the nonholonomic Hamilton–Jacobi equation:*

$$H \circ \gamma = E \, ,$$

where E *is a constant.*

Definition

Let $h\colon \mathrm{T}^*Q \to \mathbb{R}$ be a Hamiltonian function and $D \subseteq \mathrm{T}Q$ a nonholonomic distribution. A hybrid system $(\mathrm{T}^*Q, X_H^{\mathrm{nh}}, S, \Delta)$ is called a **nonholonomic hybrid system** and denoted by $\mathscr{H}_{\mathrm{nh}}$.

Definition

A sequence $\{\gamma_i\}_i$ of one-forms $\gamma_i \in \Omega^1(U_k)$ is called a **solution of the hybrid Hamilton–Jacobi problem for** $\mathscr{H}_{\mathrm{nh}}$ if, for each index i,

1. $\operatorname{Im}\gamma_i \subset C = \flat_g(D)$,
2. $\mathrm{d}\gamma_i(v, w) = 0$ for each $v, w \in \Gamma(D)$,
3. γ_i is a solution of the nonholonomic HJ equation, namely,

$$H \circ \gamma_i = E_i \,;$$

4. the compatibility condition is satisfied:

$$\operatorname{Im}(\Delta \circ \gamma_i) \subset \operatorname{Im}\gamma_{i+1} \,.$$

Theorem (Colombo, de León, Eyrea Irazú, and L. G., 2024)

Consider a hybrid nonholonomic system $\mathscr{H}_{\mathrm{nh}} = (\mathsf{T}^*Q, X_H^{\mathrm{nh}}, S, \Delta)$ *with underlying nonholonomic Hamiltonian system* (Q, H, C)*. Let* $\{\gamma_i\}_i$ *be a sequence of one-forms* $\gamma_k \in \Omega^1(U_k)$ *such that* $\operatorname{Im}\gamma_k \subset C$ *and* $\mathrm{d}\gamma_k(v, w) = 0$ *for each* $v, w \in \Gamma(D)$*. Then, the following statements are equivalent:*

1. *The sequence* $\{\gamma_i\}_i$ *is a solution of the hybrid HJ equation for* $\mathscr{H}_{\mathrm{nh}}$*.*
2. *For every continuous and piecewise curve* $c\colon \mathbb{R} \to Q$ *such that*
   1. c *intersects* $\pi_Q(S)$ *at* $\{t_k\}_k$*,*
   2. c *satisfies the equations*

$$\begin{aligned} \dot{c}(t) &= \mathsf{T}\pi_Q \circ X_H^{\mathrm{nh}} \circ \gamma_k \circ c(t), \qquad t_k < t < t_{k+1}, \\ \gamma_{k+1} \circ c(t_{k+1}) &= \Delta \circ \gamma_k \circ c(t_{k+1}), \end{aligned}$$

   then the curve $\tilde{c}\colon \mathbb{R} \to C$ *given by* $\tilde{c}(t) = \gamma_k \circ c(t)$ *for* $t \in [t_k, t_{k+1})$ *is an integral curve of the hybrid dynamics.*

Example: the generalized rigid body

- Consider a mechanical system with a Lie group as configuration space, namely $Q = G$.
- Let $\mathfrak{g}$ denote the Lie algebra of G and $\mathfrak{g}^*$ its dual.
- Its Lagrangian is the left-invariant function $L \colon \mathrm{T}G \simeq G \times \mathfrak{g} \to \mathbb{R}$ given by $L(g, v_g) = \ell(g^{-1} v_g)$, where $\ell \colon \mathfrak{g} \to \mathbb{R}$ is the reduced Lagrangian, defined by

$$\ell(\xi) = \frac{1}{2} I_{ij} \xi^i \xi^j \,,$$

for $\xi = (\xi^1, \ldots, \xi^n) \in \mathfrak{g}$, where I_{ij} are the components of the (positive-definite and symmetric) inertia tensor $\mathbb{I} \colon \mathfrak{g} \to \mathfrak{g}^*$.

Example: the generalized rigid body

- The Hamiltonian function $H\colon G \times \mathfrak{g}^* \to \mathbb{R}$ is

$$H = \frac{1}{2} I^{ij} \eta_i \eta_j \,,$$

  where I^{ij} are the components of the inverse of $\mathbb{I}$, and $\eta = (\eta_1, \ldots, \eta_n) \in \mathfrak{g}^*$.
- The constrained generalized rigid body is subject to the left-invariant nonholonomic constraint

$$D_\mu = \left\{ (g, \xi) \in G \times \mathfrak{g} \mid \langle \mu, \xi \rangle = \mu_i \xi^i = 0 \right\} \,,$$

  where $\mu = (\mu_1, \ldots, \mu_n)$ is a fixed element of $\mathfrak{g}^*$ and $\langle \cdot, \cdot \rangle$ denotes the natural pairing between a Lie algebra and its dual.

Example: the generalized rigid body

- The associated codistribution is

$$C_\mu = \left\{(g, \eta) \in G \times \mathfrak{g}^* \mid \eta_i I^{ij} \mu_j = 0\right\}.$$

- A solution of the nonholonomic HJ problem is a one-form $\gamma \colon G \to G \times \mathfrak{g}^*,\ g \mapsto (g, \gamma_1(g), \ldots, \gamma_n(g))$ satisfying

$$\begin{aligned} H \circ \gamma = \frac{1}{2} I^{ij} \gamma_i \gamma_j = E\,, \\ I^{ij} \gamma_i \mu_j = 0\,, \\ \mathrm{d}\gamma_{|D \times D} = 0\,. \end{aligned}$$

- Hereinafter, consider the lie group $G = \mathrm{SO}(3)$.

Example: the generalized rigid body

- Let $\{e_1, e_2, e_3\}$ be the canonical basis of $\mathfrak{so}(3) \simeq \mathbb{R}^3$, whose Lie brackets are

$$[e_1, e_2] = e_3\,, \quad [e_1, e_3] = -e_2\,, \quad [e_2, e_3] = e_1\,,$$

and let $\{e^1, e^2, e^3\}$ be its dual basis.

- For simplicity's sake, assume that

$$\mathbb{I} = I e^1 \otimes e^1 + I e^2 \otimes e^2 + I e^3 \otimes e^3\,,$$

and thus

$$H(g, \eta) = \frac{1}{2I^2}\left(\eta_1^2 + \eta_2^2 + \eta_3^2\right)\,.$$

Example: the generalized rigid body

- The nonholonomic distribution is given by

$$\begin{aligned}\mathcal{D}_\mu &= \left\{(g,\xi) \in \mathrm{SO}(3) \times \mathfrak{so}(3) \mid \mu_i \xi^i = 0\right\} \\ &= \langle \{\mu_2 e_1 - \mu_1 e_2,\ \mu_3 e_1 - \mu_1 e_3\} \rangle \ .\end{aligned}$$

- A solution of the HJ problem is given by

$$\gamma = \lambda_1 e^1 + \frac{\mu_3 \lambda_2 - \mu_1 \mu_2 \lambda_1}{\mu_2^2 + \mu_3^2} e^2 + \frac{\mu_2 \lambda_2 - \mu_1 \mu_3 \lambda_1}{\mu_2^2 + \mu_3^2} e^3 \,,$$

where $\lambda_2 = \pm\sqrt{2EI^2\left(\mu_2^2 + \mu_3^2\right) - \lambda_1^2\left(\mu_1^2 + \mu_2^2 + \mu_3^2\right)}$.

- The Euler angles (α, β, φ) can be used as a coordinate system for $\mathrm{SO}(3)$.

Example: the generalized rigid body

- The switching surface is the codimension-1 submanifold S of $\mathrm{SO}(3) \times \mathfrak{so}(3)^*$ given by

$$S = \{(\alpha, \beta, \varphi, \eta_1, \eta_2, \eta_3) \in \mathrm{SO}(3) \times \mathfrak{so}(3)^* \mid \alpha = 0\} \, .$$

- The impact map $\Delta \colon S \to \mathrm{SO}(3) \times \mathfrak{so}(3)^*$ is

$$\Delta \colon \, (0, \beta, \varphi, \eta_1, \eta_2, \eta_3) \mapsto (0, \beta, \varphi, \varepsilon\eta_1, \eta_2, \eta_3) \, ,$$

  for s constant ε.

- Let γ^- and γ^+ denote the solutions to the Hamilton–Jacobi equation before and after the impact, respectively, where

$$\gamma^\pm = \lambda_1^\pm e^1 + \frac{\mu_3\lambda_2^\pm - \mu_1\mu_2\lambda_1^\pm}{\mu_2^2 + \mu_3^2} e^2 + \frac{\mu_2\lambda_2^\pm - \mu_1\mu_3\lambda_1^\pm}{\mu_2^2 + \mu_3^2} e^3 \, .$$

Example: the generalized rigid body

- Then,

$$\lambda_1^+ = \varepsilon \lambda_1^- ,$$
$$\lambda_2^+ = \lambda_2^- + (\varepsilon - 1) \frac{\mu_1 \mu_2}{\mu_3} \lambda_1^- ,$$
$$\lambda_2^+ = \lambda_2^- + (\varepsilon - 1) \frac{\mu_1 \mu_3}{\mu_2} \lambda_1^- ,$$

which has solutions if $\mu_3 = \pm \mu_2$ or if $\varepsilon = 1$.

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Thanks for your kind attention!