A first contact with contact geometry

Asier López-Gordón*,

Instituto de Ciencias Matemáticas (ICMAT),
Consejo Superior de Investigaciones Científicas, Madrid, Spain.

May 30, 2024

These are the notes I used for a talk I gave at the PhD students' seminar from the Faculty of Mathematics at the Complutense University of Madrid. These notes are not a comprehensive exposition of the topics covered; for detailed information, readers should consult the references.

1 Contact transformations

Definition 1. (Lie) A **contact element** (or **line element**) of $\mathbb{R}^2$ is a point $(x, z) \in \mathbb{R}^2$ and a line passing through that point. If the slope p of this line is finite, then the equation for the line can be written as

$$\mathrm{d}z - p\mathrm{d}x = 0\,, \tag{1}$$

and the space of contact elements on $\mathbb{R}^2$ can be identified with $\mathbb{R}^3$ with coordinates (x, p, z).

Similarly, a **contact element** of $\mathbb{R}^n \times \mathbb{R}$ is a hyperplane passing through a point $(x_1, \ldots, x_n, z) \in \mathbb{R}^n \times \mathbb{R}$ defined by the equation

$$\mathrm{d}z - \sum_{i=1}^{n} p_i \mathrm{d}x^i = 0\,. \tag{2}$$

The space of contact elements on $\mathbb{R}^n \times \mathbb{R}$ can be identified with $\mathbb{R}^{2n+1}$ with coordinates (x^i, p_i, z), $i \in \{1, \ldots, n\}$.

Consider the ordinary differential equation

$$F\big(x, z(x), z'(x)\big) = 0\,. \tag{3}$$

A solution $z = z(x)$ of (3) corresponds to an integral curve $x \mapsto \big(x, z(x), z'(x)\big)$ of the plane field given by (1). More generally, a solution

$$(x^1, \ldots, x^n) \mapsto \left(x^1, \ldots, x^n, z(x^1, \ldots, x^n), \frac{\partial z}{\partial x^1}, \ldots, \frac{\partial z}{\partial x^n}\right)$$

of the partial differential equation

$$F\left(x^1, \ldots, x^n, z(x^1, \ldots, x^n), \frac{\partial z}{\partial x^1}, \ldots, \frac{\partial z}{\partial x^n}\right) = 0 \tag{4}$$

corresponds to an integral submanifold of the hyperplane field on $\mathbb{R}^{2n+1}$ given by (2).

*e-mail: asier.lopez@icmat.es

Definition 2. A transformation $(x^i, p_i, z) \mapsto (\tilde{x}^i, \tilde{p}_i, \tilde{z})$ of $\mathbb{R}^{2n+1}$ is called a **contact transformation** if

$$\mathrm{d}\tilde{z} - \sum_{i=1}^{n} \tilde{p}_i \mathrm{d}\tilde{x}^i = \rho \left(\mathrm{d}z - \sum_{i=1}^{n} p_i \mathrm{d}x^i \right) ,$$

for some function $\rho \colon \mathbb{R}^{2n+1} \to \mathbb{R} \setminus \{0\}$.

Contact transformations carry solutions of (4) into solutions of the transformed equation

$$\tilde{F} \left(\tilde{x}^1, \ldots, \tilde{x}^n, \tilde{z}(\tilde{x}^1, \ldots, \tilde{x}^n), \frac{\partial \tilde{z}}{\partial \tilde{x}^1}, \frac{\partial \tilde{z}}{\partial \tilde{x}^n} \right) = 0 .$$

For more details on the use of contact transformations for studying differential equations see [13, 18].

2 Basics on differential geometry and contact geometry

Let M be an n-dimensional differentiable manifold. Recall that the tangent bundle of M is the space

$$\mathsf{T}M = \bigsqcup_{x \in M} \mathsf{T}_x M ,$$

with the projection

$$\tau_M \colon \mathsf{T}M \ni (x, v) \mapsto x \in M .$$

A rank k distribution D on M is a smooth assignment of a rank k vector subspace $D_x \subseteq \mathsf{T}_x M$ for each $x \in M$. It is a vector bundle with the projection $\tau_M|_D \colon D \to M$.

Theorem 1 (Frobenius)**.** *Let D be a distribution on M. Then, the following statements are equivalent:*

1. *For every $x \in M$, there exists a submanifold $N \subseteq M$ such that $D_x = \mathsf{T}_x N$ (i.e., D is **integrable**).*

2. *For each pair of vector fields $X, Y \in \mathfrak{X}(M)$ such that $X(x), Y(x) \in D_x$ for all $x \in M$ we have that $[X, Y](x) \in D_x$ (i.e., D is **involutive**).*

A corank-1 vector subspace W from a real vector space V can be expressed as the kernel of a covector, namely, $W = \ker \alpha$ for some $\alpha \in V^*$. Consequently, a corank-1 distribution D on a manifold M (i.e., a field of tangent hyperplanes on M) can be locally expressed as the kernel of a local one-form, namely, $D_x = \ker \alpha_x$ for $x \in U \subseteq M$ and $\alpha \in \Omega^1(U)$, with U an open subset of M.

Let us recall that the **exterior product** (or **wedge product**) of a p-form $\alpha \in \Omega^p(M)$ and a q-form $\beta \in \Omega^q(M)$ is a $(p+q)$-form

$$\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha \in \Omega^{p+q}(M) .$$

Proposition 2. *Let D be a corank-1 distribution on a manifold M. Suppose that D is locally given by the kernel of a local one-form α. Then, D is integrable if and only if*

$$\alpha \wedge \mathrm{d}\alpha = 0 .$$

Example 1. Consider $M = \mathbb{R}^3$ with canonical coordinates (x, y, z) and cylindrical coordinates (r, ϕ, z). The distribution $D = \ker\alpha$ for $\alpha = y\mathrm{d}x - x\mathrm{d}y$ is integrable. Indeed,

$$D = \left\langle x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle = \left\langle r\frac{\partial}{\partial r}, \frac{\partial}{\partial z} \right\rangle ,$$

and thus $D_p = \mathsf{T}_p N$ for all $p \in N$, where

$$N = \left\{(r, \phi, z) \in \mathbb{R}^3 \mid \phi = \text{const.}\right\} .$$

A distribution is called a **contact distribution** if it is "as far as possible" from being integrable. More specifically, it is defined as follows.

Definition 3. Let M be a $(2n+1)$-dimensional manifold. A **contact distribution** ξ on M is a corank-1 distribution locally given by $\xi = \ker\alpha$ such that

$$\alpha \wedge (\mathrm{d}\alpha)^n = \alpha \wedge \underbrace{\mathrm{d}\alpha \wedge \cdots \wedge \mathrm{d}\alpha}_{n \text{ times}} \neq 0 .$$

If there exists a global one-form such that $\xi = \ker\alpha$, then it is called a **contact form** and ξ is called **co-orientable**. A pair (M, ξ) (respectively, (M, α)) is called a **contact manifold** (respectively, **co-oriented contact manifold**).

Observe that a co-orientable contact distribution ξ on M defines an equivalence class of contact forms

$$\alpha \sim \beta \iff \ker\alpha = \ker\beta = \xi \iff \exists f \colon M \to \mathbb{R}\setminus\{0\} \text{ such that } \alpha = f\beta .$$

These equivalence classes are called **conformal classes**, and two contact forms belonging to the same conformal class are said to be **conformal** to each other.

Example 2. Any odd-dimensional Euclidean space $M = \mathbb{R}^{2n+1}$ (where $n \geq 1$) with canonical coordinates $(x^1, \ldots, x^n, p_1, \ldots, p_n, z)$ has a canonical contact form

$$\alpha = \mathrm{d}z - p_i \mathrm{d}x^i . \tag{5}$$

Example 3. Consider $\mathbb{R}^{2n}$ with canonical coordinates $(x^1, \ldots, x^n, y_1, \ldots, y_n)$. Then, the $(n-1)$-sphere

$$\mathbb{S}^{2n-1} = \left\{x \in \mathbb{R}^{2n+1} \mid \|x\| = 1\right\}$$

is endowed with a contact form

$$\alpha = x^i \mathrm{d}y_i - y_i \mathrm{d}x^i .$$

Example 4 (Not co-orientable contact distribution)**.** Let $M = \mathbb{R}^{n+1} \times \mathbb{RP}^n$. Denote by $(x^0, \ldots x^n)$ the coordinates in $\mathbb{R}^{n+1}$ and by $[y_0 \colon \cdots \colon y_n]$ the homogeneous coordinates in $\mathbb{RP}^n$. Then,

$$\xi = \ker \sum_{\mu=0}^{n} y_\mu \mathrm{d}x^\mu$$

is a contact distribution on M. Indeed, the one-form $y_\mu \mathrm{d}x^\mu$ is well-defined up to a scaling by a non-zero real constant. On each open subset $U_\mu = \{y_\mu \neq 0\} \subset M$, we have the contact form

$$\alpha_\mu = \mathrm{d}x^\mu + \sum_{\nu \neq \mu} \frac{y_\nu}{y_\mu} \mathrm{d}x^\nu .$$

If n is even, then M is not orientable. This implies that there can be no volume form on M, so in particular there exists no global contact form for ξ.

By a different argument, one can show that for n odd the contact distribution ξ is also not co-orientable (see Proposition 2.1.13 in [12]).

Theorem 3 (Darboux)**.** *Let (M, α) be a $(2n+1)$-dimensional co-oriented contact manifold. Then, around each point $x \in M$ there is a chart $(U; x^i, p_i, z)$ such that α is written (5).*

A contact distribution can be regarded as an atlas for the manifold M whose coordinate changes are local contact transformations. This was the viewpoint until the 1960's.

Proposition 4. *Let M be a $(2n+1)$-dimensional smooth manifold. A one-form α on M is a contact form if and only if the map*

$$\begin{aligned}\flat_\alpha \colon \mathsf{T}_x M &\to (\mathsf{T}_x M)^* \\ v &\mapsto \alpha_x(v)\alpha + \mathrm{d}\alpha_x(v, \cdot)\end{aligned}$$

is an isomorphism of vector spaces.

This implies that we have a decomposition

$$\mathsf{T}_x M = \ker \alpha_x \oplus \ker \mathrm{d}\alpha_x = \xi_x \oplus \ker \mathrm{d}\alpha_x\,, \quad \forall x \in M\,.$$

Clearly, this decomposition depends on the choice of contact form.

Definition 4. Let (M, α) be a co-oriented contact manifold. The **Reeb vector field** is the unique vector field $R \in \mathfrak{X}(M)$ given by

$$R = \flat_\alpha^{-1}(\alpha)\,,$$

or, equivalently,

$$R \in \ker \mathrm{d}\alpha \quad \text{and} \quad \alpha(R) = 1\,.$$

In Darboux coordinates,

$$R = \frac{\partial}{\partial z}\,.$$

The decomposition above can also be written as

$$\mathsf{T}_x M = \xi_x \oplus \langle R_x \rangle\,.$$

Definition 5. Let (M, ξ) be a $(2n+1)$ dimensional contact manifold. A submanifold N of M is called **isotropic** if it is an integral submanifold of ξ, that is, if $\mathsf{T}_x N = \xi_x$ for all $x \in N$. In addition, if N is of maximal dimension (i.e., $\dim N = n$), it is called **Legendrian**.

3 On the existence of contact distributions

A natural question one may ask is: given an odd-dimensional manifold M, there exists any contact distribution on M? Essentially, the answer is "yes" for dimension 3, and "not necessarily" for higher dimensions.

Theorem 5 (Martinet)**.** *Every compact orientable 3-manifold (without boundary) admits a contact distribution.*

Theorem 6 (Stong)**.** *The compact orientable manifold $\mathrm{SU}(3)/\mathrm{SO}(4) \times \mathbb{S}^{2n-4}$, for $n \geq 2$, admits no contact distributions.*

4 Contact Hamiltonian systems and dissipative mechanics

Definition 6. Let (M, α) be a co-oriented contact manifold and let $f \in \mathscr{C}^\infty(M)$. The **Hamiltonian vector field of** f is given by

$$X_f = \flat_\alpha^{-1}(\mathrm{d}f) - (R(f) + f)R\,,$$

or, equivalently,

$$\alpha(X_f) = -f\,, \quad \mathrm{d}\alpha(X_f, \cdot) = \mathrm{d}f - R(f)\alpha\,.$$

In Darboux coordinates (q^i, p_i, z),

$$X_f = \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} - \left(\frac{\partial f}{\partial q^i} + p_i\frac{\partial f}{\partial z}\right)\frac{\partial}{\partial p_i} + \left(p_i\frac{\partial f}{\partial p_i} - f\right)\frac{\partial}{\partial z}\,.$$

In particular, observe that the Reeb vector field is the Hamiltonian vector field of $f \equiv -1$. Conversely, if X_f is the Hamiltonian vector field of f with respect to α then, on the open subset $U = M \setminus f^{-1}(0)$, it is the Reeb vector field associated with the conformal contact form $\tilde{\alpha} = -\frac{1}{f}\alpha$. On the zero level set of f the Hamiltonian vector field X_f may be written as the reparametrization of the Liouville vector field associated with the exact symplectic form induced by the contact form (refer to [3] for more details).

Definition 7. A **contact Hamiltonian system** is a triple (M, α, h) formed by a co-oriented contact manifold (M, α) and a function $h \in \mathscr{C}^\infty(M)$ called the **Hamiltonian function**.

Physically, one can regard h as the total energy of the system. The dynamics of (M, α, h) are given by the integral curves of X_h. In Darboux coordinates, an integral curve $c\colon I \subseteq \mathbb{R} \to M$, $c(t) = (q^i(t), p_i(t), z(t))$ of X_h satisfies the **contact Hamilton equations**:

$$\begin{aligned}
\frac{\mathrm{d}q^i}{\mathrm{d}t}(t) &= \frac{\partial h}{\partial p_i} \circ c(t)\,,\\
\frac{\mathrm{d}p_i}{\mathrm{d}t}(t) &= -\frac{\partial h}{\partial q^i} \circ c(t) - p_i(t)\frac{\partial h}{\partial z} \circ c(t)\,,\\
\frac{\mathrm{d}z}{\mathrm{d}t}(t) &= p_i(t)\frac{\partial h}{\partial p_i} \circ c(t) - h \circ c(t)\,.
\end{aligned}$$

These equations resemble the classical Hamilton equations. Indeed, the first of the equations is identical. The second has an extra term on the right-hand side accounting for the dependence of h on z. Note that the right-hand side of the third equation is like the "Lagrangian". As a matter of fact, contact Hamiltonian systems have a Lagrangian counterpart, where Lagrangian functions depend on the action functional.

Very loosely, the Herglotz functional $\mathcal{A}$ is like the usual action functional, but instead of being given by an integral is given by the ODE

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}\big[c(t)\big] = L\Big(c(t), \dot{q}(t), \mathcal{A}\big[c(t)\big]\Big)\,.$$

One seeks for curves $c\colon I \subseteq \mathbb{R} \to Q$ that are critical points of $\mathcal{A}$.

Fixing two points q_1 and q_2 in Q, and an interval $[a, b]$, the **path space from** q_1 **to** q_2 is the following set of curves:

$$\Omega(q_1, q_2, [a, b]) = \left\{c \in \mathscr{C}^2\big([a, b] \to Q\big) \mid c(a) = q_1,\ c(b) = q_2\right\}\,.$$

It can be proven that $\Omega(q_1, q_2, [a, b])$ is an infinite-dimensional smooth manifold. Its tangent space $\mathsf{T}_c\Omega(q_1, q_2, [a, b])$ at c is the set of maps $v \in \mathscr{C}^2\big([a, b] \to \mathsf{T}Q\big)$ such that $\tau_q \circ v = c$ and

$v(a) = v(b) = 0$. A tangent vector $v \in \mathsf{T}_c\Omega(q_1, q_2, [a,b])$ is called an **infinitesimal variation of the curve c subject to fixed endpoints**.

Consider the operator

$$\mathcal{Z}\colon \Omega\big(q_1, q_2, [a,b]\big) \to \mathscr{C}^2\big([a,b] \to \mathbb{R}\big)$$

that assigns to each curve $c \in \Omega(q_1, q_2, [a,b])$ the function $\mathcal{Z}(c)$ that is the solution of the following Cauchy problem:

$$\begin{aligned}
&\frac{\mathrm{d}\mathcal{Z}(c)(t)}{\mathrm{d}t} = L\big(c(t), \dot{c}(t), \mathcal{Z}(c)(t)\big)\,,\\
&\mathcal{Z}(c)(a) = z_a\,.
\end{aligned}$$

The quantity $Z(c)(t)$ can be interpreted as the action of the curve c at time t. The **Herglotz action functional** is the map $\mathcal{A}\colon \Omega(q_1, q_2, [a,b]) \to \mathbb{R}$ that assigns to each curve the solution of the Cauchy problem above evaluated at the endpoint, namely,

$$\mathcal{A}\colon c \mapsto \mathcal{Z}(c)(b)\,.$$

The trajectories of the dynamical system described by (Q, L) are given by the following variational principle. A curve $c \in \Omega(q_1, q_2, [a,b])$ satisfies the **Herglotz variational principle** if it is a critical point of the Herglotz action functional, that is,

$$\mathrm{d}\mathcal{A}(c) = 0\,.$$

A curve $c \in \Omega(q_1, q_2, [a,b])$ is a critical point of $\mathcal{A}$ if and only if it satisfies the **Herglotz–Euler–Lagrange equations**:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial v^i}\big(c(t), \dot{c}(t), \mathcal{Z}(c)(t)\big) - \frac{\partial L}{\partial q^i}\big(c(t), \dot{c}(t), \mathcal{Z}(c)(t)\big)\\
&\qquad - \frac{\partial L}{\partial v^i}\big(c(t), \dot{c}(t), \mathcal{Z}(c)(t)\big)\frac{\partial L}{\partial z}\big(c(t), \dot{c}(t), \mathcal{Z}(c)(t)\big) = 0\,.
\end{aligned}$$

Refer to [10, 15] for more details on the Herglotz variational principle.

Example 5 (The harmonic oscillator with linear damping)**.** Consider the solution $x\colon \mathbb{R} \to \mathbb{R}$ of the second-order ordinary differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}(t) = -x(t) - \kappa\frac{\mathrm{d}x}{\mathrm{d}t}(t)\,, \tag{6}$$

where $\kappa \in \mathbb{R}$. Defining $p = \mathrm{d}x/\mathrm{d}t$, we can reduce it to the system of first-order ordinary differential equations

$$\begin{aligned}
&\frac{\mathrm{d}x}{\mathrm{d}t}(t) = p(t)\,,\\
&\frac{\mathrm{d}p}{\mathrm{d}t}(t) = -x(t) - \kappa p(t)\,.
\end{aligned}$$

We can obtain this system as the two first contact Hamilton equations from the contact Hamilton system $(\mathbb{R}^3, \alpha, h)$, where $\alpha = \mathrm{d}z - p\mathrm{d}x$ and

$$h = \frac{p^2}{2} + \frac{x^2}{2} + \kappa z\,.$$

Equivalently, we can obtain (6) as the Herglotz–Euler–Lagrange equation for the action-dependent Lagrangian function

$$L(x, v, z) = \frac{v^2}{2} - \frac{x^2}{2} - \kappa z\,.$$

Remark 1. The Hamiltonian function is not conserved, but dissipated in a certain manner, namely,

$$X_h(h) = -R(h)h\,.$$

Similarly, the contact distribution is preserved along the flow of X_h, but the contact distribution is not, namely,

$$\mathscr{L}_{X_h}\alpha = -R(h)\alpha\,.$$

Intuitively, the Lie derivative $\mathscr{L}_X\beta$ of a differential form β with respect to a vector field X is the infinitesimal transformation resulting from performing an infinitesimal displacement of β along the direction of X. More precisely, if ϕ_t denotes the flow of X, we have

$$\mathscr{L}_X\beta = \lim_{t\to 0}\frac{\phi_t^*\beta - \beta}{t} = \lim_{t\to 0}\frac{(\beta_i\circ\phi_t)\,\mathrm{d}(x^i\circ\phi_t) - \beta_i\mathrm{d}x^i}{t}\,,$$

where $\beta = \beta_i\mathrm{d}x^i$.

Definition 8. Let (M,α,h) be a contact Hamiltonian system. A **dissipated quantity** (with respect to (M,α,h)) is a solution $f\in\mathscr{C}^\infty(M)$ to the partial differential equation

$$X_h(f) = -R(h)f\,.$$

In particular, h is a dissipated quantity with respect to (M,α,h).

Definition 9. Let (M,α) be a co-oriented contact manifold. The **Jacobi bracket** is the map $\{\cdot,\cdot\}\colon\mathscr{C}^\infty(M)\times\mathscr{C}^\infty(M)\to\mathscr{C}^\infty(M)$ given by

$$\{f,g\} = X_f(g) + gR(f)\,.$$

The Jacobi bracket satisfies the following properties:

1. $\mathbb{R}$-bilinearity,
2. skew-symmetry,
3. the Jacobi identity: $\{f,\{g,h\}\}+\{g,\{h,f\}\}+\{h,\{f,g\}\}=0$,

$\therefore$ it is a Lie bracket,

4. the weak Leibniz rule: $\{f,gh\} = \{f,g\}h + \{f,h\}g - ghR(f)$.

Proposition 7. *Let (M,α,h) be a contact Hamiltonian system. Then, a function $f\in\mathscr{C}^\infty(M)$ is a dissipated quantity if and only if $\{f,h\}=0$.*

It is worth remarking that, unlike in the case of Poisson brackets, $\{f,h\}=0$ does not imply X_f is tangent to the level sets of g. In other words, the submanifolds $g^{-1}(\lambda)$ will, in general, no longer be invariant by the flow of X_f.

Definition 10. A **completely integrable contact system** is a triple (M,α,F), where (M,α) is a co-oriented contact manifold and

$$F = (f_0,\dots,f_n)\colon M\to\mathbb{R}^{n+1}$$

is a map such that $\{f_\alpha,f_\beta\}=0$ for all $\alpha,\beta\in\{0,\dots,n\}$ and $\operatorname{rank}\mathsf{T}_xF\geq n$ for all $x\in M$.

For each $\Lambda \in \mathbb{R}^{n+1} \setminus \{0\}$, let $\langle \Lambda \rangle_+$ denote the ray generated by Λ, namely,

$$\langle \Lambda \rangle_+ = \{x \in \mathbb{R}^{n+1} \mid \exists\, r \in \mathbb{R}_+ \colon x = r\Lambda\} .$$

Consider the subset

$$M_{\langle \Lambda \rangle_+} = F^{-1}\big(\langle \Lambda \rangle_+\big) = \{x \in M \mid \exists\, r \in \mathbb{R}_+ \colon F(x) = r\Lambda\}$$

Theorem 8 (Colombo, de León, Lainz, L. G., 2023)**.** *Let (M, α, F) be a completely integrable contact system, where $F = (f_0, \dots, f_n)$, and let $\Lambda \in \mathbb{R}^{n+1} \setminus \{0\}$. Assume that the Hamiltonian vector fields $X_{f_0}, \dots, X_{f_n}$ are complete. Then:*

1. *The submanifold $M_{\langle \Lambda \rangle_+}$ is invariant by the flows of $X_{f_0}, \dots, X_{f_n}$, and diffeomorphic to $\mathbb{T}^k \times \mathbb{R}^{n+1-k}$ for some $k \leq n$.*
2. *There exist coordinates $(y^0, \dots, y^n, \tilde{A}_1, \dots, \tilde{A}_n)$ on U such that the integral curves of X_{f_β} are given by*

$$\begin{aligned} \dot{y}^\alpha &= \Omega^\alpha(\tilde{A}_1, \dots, \tilde{A}_n)\,, \qquad && \alpha = 0, \dots, n\,, \\ \dot{\tilde{A}}_i &= 0\,, && i = 1, \dots, n\,. \end{aligned}$$

5 Thermodynamics

"*Every mathematician knows that it is impossible to understand any elementary course in thermodynamics. The reason is that thermodynamics is based on a rather complicated mathematical theory, on contact geometry.*" V. I. Arnol'd.

Consider a thermodynamical system with energy U, temperature T, entropy S, pressure P and volume V. Then, the states of equilibrium are determined by

$$\mathrm{d}U = T\mathrm{d}S - P\mathrm{d}V\,,$$

together with the equations of state (which depend on the substance considered). We can think of a 5-manifold with coordinates (U, T, S, P, V) and contact form

$$\alpha = \mathrm{d}U - T\mathrm{d}S + P\mathrm{d}V\,,$$

and regard equilibrium states as Legendrian submanifolds (of dimension 2).

Example 6 (Ideal gas with constant number of particles)**.** The equations of state are

$$PV = k_1 T\,, \quad U = k_2 T\,,$$

for some constants k_1 and k_2.

More generally, we could have k different types of substances with chemical potentials $\mu_1, \dots, \mu_k$ and $N_1, \dots, N_k$ numbers of particles. In that case, the thermodynamical variables form a $5 + 2k$ manifold with a contact form

$$\alpha = \mathrm{d}U - T\mathrm{d}S + P\mathrm{d}V + \sum_{i=1}^{k} \mu_i \mathrm{d}N_i\,.$$

Generally, we will have an odd-dimensional manifold whose coordinates are the pairs of conjugate variables (temperature-entropy, pressure-volume, and so forth) together with the internal energy.

6 Turing complete Euler flows and Reeb vector fields

This section is a very imprecise overview of a series of papers by Cardona, Miranda, Peralta–Salas and Presas [5–7].

Let (M, g) be a Riemannian manifold. The stationary Euler equations on (M, g) read

$$\nabla_X X = -\nabla p\,, \quad \operatorname{div} X = 0\,,$$

where p stands for the hydrodynamic pressure and X for the velocity field of the fluid. A vector field X on M is called **Eulerisable** if there exists a Riemannian metric g on M such that X satisfies the Euler equation for (M, g).

A vector field X on an odd-dimensional Riemannian manifold (M, g) is **Beltrami** if

$$\operatorname{div} X = 0\,, \quad \operatorname{curl} X = fX\,,$$

for some $f \in \mathscr{C}^\infty(M)$. Beltrami fields are stationary solutions of the Euler equations with constant Bernouilli function

$$B \coloneqq p + \frac{1}{2} g(X, X)\,.$$

Theorem 9. *Let (M, g) be an odd-dimensonal Riemannian manifold. Assume that X is a nowhere-vanishing Beltrami field on M such that*

$$\operatorname{curl} X = fX\,, \quad f > 0\,.$$

Then, any smooth, nonsingular rotational Beltrami field on M is the Reeb vector field of some contact form α on M.

Roughly speaking, a vector field $X \in \mathfrak{X}(M)$ is Turing complete if, given a Turing machine T, one can construct a point $p \in M$ and an open subset $U \subset M$ such that the trajectory of X through p intersects U if and only if T halts.

From Turing machines to undecibility of Euler flows:

1. A **generalized shift** is a map $\phi\colon \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ for a finite alphabet $\mathcal{A} = \{0, 1\}$. They can be used to simulate any Turing machine [Moore, 1991].

2. Generalized shifts can be understood as maps of the **square Cantor set** $C^2 \coloneqq C \times C \subset [0, 1] \times [0, 1]$, where C is the standard Cantor ternary set in the unit interval $[0, 1]$.

3. Any bijective generalized shift, understood as a map $\varphi\colon C^2 \to C^2$, can be extended as an area-preserving diffeomorphism of the disk $\varphi\colon D \to D$ which is the identity in a neighbourhood of ∂D.

4. Let (M, ξ) be a co-orientable contact 3-manifold and $\varphi\colon D \to D$ an area-preserving diffeomorphism such that $\varphi|_{\partial D} = \mathrm{id}$. Then, there is a contact form α whose Reeb vector field R is associated to φ. (More precisely, there is a Poincaré section of R whose first-return map is conjugate to φ.)

5. In particular, this implies that there exists a contact form α on (M, ξ) whose Reeb vector field R is Turing complete.

$\therefore$ There exists a Eulerisable vector field on $\mathbb{S}^3$ that is Turing complete.

References

[1] A. Anahory Simoes, M. de León, M. Lainz Valcázar, and D. Martín de Diego. "Contact Geometry for Simple Thermodynamical Systems with Friction". *Proc. R. Soc. A.*, **476**(2241), p. 20200244 (2020). ISSN: 1364-5021, 1471-2946. DOI: 10.1098/rspa.2020.0244.

[2] V. I. Arnol'd. "Contact Geometry: The Geometrical Method of Gibbs's Thermodynamics". In: *Proceedings of the Gibbs Symposium (New Haven, CT, 1989)*. Amer. Math. Soc., Providence, RI, 1990, pp. 163–179. ISBN: 978-0-8218-0157-4.

[3] A. Bravetti, M. de León, J. C. Marrero, and E. Padrón. "Invariant Measures for Contact Hamiltonian Systems: Symplectic Sandwiches with Contact Bread". *J. Phys. A: Math. Theor.*, **53**(45), p. 455205 (2020). ISSN: 1751-8121. DOI: 10.1088/1751-8121/abbaaa.

[4] A. Bravetti. "Contact Geometry and Thermodynamics". *Int. J. Geom. Methods Mod. Phys.*, **16**, p. 1940003 supp01 (2019). ISSN: 0219-8878, 1793-6977. DOI: 10.1142/S0219887819400036.

[5] R. Cardona, E. Miranda, and D. Peralta-Salas. "Looking at Euler Flows through a Contact Mirror: Universality and Undecidability". In: *European Congress of Mathematics*. EMS Press, Berlin, 2023, pp. 367–393. ISBN: 978-3-9854705-1-8 978-3-9854755-1-3.

[6] R. Cardona, E. Miranda, D. Peralta-Salas, and F. Presas. "Constructing Turing Complete Euler Flows in Dimension 3". *Proc. Natl. Acad. Sci. U.S.A.*, **118**(19), e2026818118 (2021). ISSN: 0027-8424, 1091-6490. DOI: 10.1073/pnas.2026818118.

[7] R. Cardona, E. Miranda, D. Peralta-Salas, and F. Presas. "Universality of Euler Flows and Flexibility of Reeb Embeddings". *Advances in Mathematics*, **428**, p. 109142 (2023). ISSN: 0001-8708. DOI: 10.1016/j.aim.2023.109142.

[8] L. Colombo, M. de León, M. Lainz, and A. López-Gordón. *Liouville-Arnold Theorem for Contact Hamiltonian Systems*. 2023. arXiv: 2302.12061 [math.SG].

[9] M. de León and M. Lainz Valcázar. "Contact Hamiltonian Systems". *J. Math. Phys.*, **60**(10), p. 102902 (2019). ISSN: 0022-2488. DOI: 10.1063/1.5096475.

[10] M. de León and M. Lainz Valcázar. "Singular Lagrangians and Precontact Hamiltonian Systems". *Int. J. Geom. Methods Mod. Phys.*, **16**(10), p. 1950158 (2019). ISSN: 0219-8878. DOI: 10.1142/S0219887819501585.

[11] H. Geiges. "A Brief History of Contact Geometry and Topology". *Expo. Math.*, **19**(1), pp. 25–53 (2001). ISSN: 0723-0869. DOI: 10.1016/S0723-0869(01)80014-1.

[12] H. Geiges. *An Introduction to Contact Topology*. Cambridge Studies in Advanced Mathematics. Cambridge University Press: Cambridge, 2008. ISBN: 978-0-521-86585-2. DOI: 10.1017/CBO9780511611438.

[13] M. Giaquinta and S. Hildebrandt. "Partial Differential Equations of First Order and Contact Transformations". In: *Calculus of Variations II*. Ed. by M. Giaquinta and S. Hildebrandt. Springer: Berlin, Heidelberg, 2004, pp. 441–604. ISBN: 978-3-662-06201-2. DOI: 10.1007/978-3-662-06201-2_4.

[14] K. Grabowska and J. Grabowski. "A Geometric Approach to Contact Hamiltonians and Contact Hamilton–Jacobi Theory". *J. Phys. A: Math. Theor.*, **55**(43), p. 435204 (2022). ISSN: 1751-8121. DOI: 10.1088/1751-8121/ac9adb.

[15] R. B. Guenther, C. M. Guenther, and J. A. Gottsch. *The Herglotz Lectures on Contact Transformations and Hamiltonian Systems*. Lecture Notes in Nonlinear Analysis. Wydaw. Uniwersytetu Mikołaja Kopernika: Toruń, 1996. ISBN: 978-83-231-0736-1.

[16] J. M. Lee. *Introduction to Smooth Manifolds*. Vol. 218. Graduate Texts in Mathematics. Springer New York: New York, NY, 2012. ISBN: 978-1-4419-9981-8 978-1-4419-9982-5. DOI: 10.1007/978-1-4419-9982-5.

[17] P. Libermann and C.-M. Marle. *Symplectic Geometry and Analytical Mechanics*. Springer Netherlands: Dordrecht, 1987. ISBN: 978-90-277-2439-7 978-94-009-3807-6. DOI: 10.1007/978-94-009-3807-6.

[18] T. Morimoto. "Monge-Ampère Equations Viewed from Contact Geometry". *Banach Center Publications*, **39**(1), pp. 105–121 (1997). ISSN: 0137-6934. URL: https://eudml.org/doc/208653.

[19] Overflowian. *Examples of Odd-Dimensional Manifolds That Do Not Admit Contact Structure*. MathOverflow, 2019. URL: https://mathoverflow.net/q/325397.

[20] R. E. Stong. "Contact Manifolds". *J. Differential Geometry*, **9**, pp. 219–238 (1974). ISSN: 0022-040X,1945-743X.